Published by Institute of Physics Publishing for SISSA
Received: *February 1, 2006*
Accepted: *March 22, 2007*
Published: *April 3, 2007*

# Planar super-Landau models revisited

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Abstract: We use the methods of $\mathcal{PT}$-symmetric quantum theory to find a one-parameter family of $ISU(1|1)$-invariant planar super-Landau models with positive norm, uncovering an 'accidental', and generically spontaneously-broken, worldline supersymmetry, with charges that have a Sugawara-type representation in terms of the $ISU(1|1)$ charges. In contrast to standard models of supersymmetric quantum mechanics, it is the norms of states rather than their energies that are parameter-dependent, and the spectrum changes discontinuously in the limit that worldline supersymmetry is restored.

Keywords: Global Symmetries, Superspaces, Supersymmetry Breaking, Integrable Equations in Physics.



 

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## 1. Introduction

In recent works, three of us have explored the mathematics and physics of 'super-Landau' models, which are quantum mechanical models for a charged particle on a homogeneous superspace, such that the 'bosonic' truncation is either Landau's original model for a charged particle moving on a plane under the influence of a uniform magnetic field, or Haldane's spherical version of it. The former case yields 'planar' super-Landau models and the latter case yields 'spherical' super-Landau models; the two are related by a limiting process in which the sphere becomes a plane as its radius is taken to infinity.

The simplest spherical super-Landau models are those for which the homogeneous superspace has isometry supergroup $SU(2|1)$, and the simplest of these is the 'superspherical' Landau model for a charged particle on the projective superspace $CP(1|1)$, which is the complex 'Riemann supersphere' and can be viewed as the coset superspace $SU(2|1)/U(1|1)$. In a limit in which only the lowest Landau level (LLL) of this model survives, it describes a fuzzy Riemann supersphere [1]. The other spherical super-Landau models with SU(2|1) symmetry are the 'superflag' Landau models, for which the homogeneous superspace is the coset superspace SU(2|1)/[U(1) × U(1)] [2]. In this case there is an additional anticommuting variable, and a corresponding 'fermionic Wess-Zumino' term with real number coefficient $M$. There is therefore a 1-parameter family of superflag Landau models, and the $M{=}0$ model turns out to be equivalent to the superspherical model.

The quantum theory of the spherical super-Landau models was worked out in [1, 2] and a number of intriguing properties were uncovered. The spherical models are conceptually simpler than the planar models because the degeneracies at each Landau level are finite, but the non-linearity of the configuration space leads to computational complexities. For this reason, it is useful to study the class of planar super-Landau models obtained as the planar limit of the spherical super-Landau models; these all have isometry supergroup $ISU(1|1)$. The planar limit of the superspherical model yields the 'superplane' Landau model, while the planar limit of the superflag Landau models yields the 'planar-superflag' Landau models [3], which are parametrized by the real number $M$, with the $M{=}0$ model being equivalent to the superplane model.

One result of [1 − 3] was that there are ghosts in all Landau levels with $N{>}2M$, and zero-norm states in all levels with $N{=}2M$ (which is possible when $2M$ is a non-negative integer). This result assumes a natural superspace norm, invariant under the superspace isometries and with respect to which the Hamiltonian is hermitian, and it shows that this norm is indefinite. This was not unexpected since the classical equations of motion for the 'fermionic' variables are (except in the LLL limit) second order in time derivatives, rather than first order; this typically leads to ghosts in quantum field theory, and in quantum mechanics [4]. However, more options are available in quantum mechanics.[1] In particular, the possibility of an alternative norm was not addressed in [1 − 3], although it is not difficult to see that there must exist a positive norm: the hermiticity of the Hamiltonian with respect to any non-degenerate norm implies that it is both diagonalizable and has real eigenvalues, and it is therefore manifestly hermitian with respect to the natural positive-definite norm in the basis in which it is diagonal. However, it is not immediately clear what the consequences are for the symmetries, nor whether there are further possibilities. One purpose of this paper is to explore the possibilities for symmetry-preserving norms that maintain the hermiticity of the Hamiltonian, and thereby to determine whether the ghosts found previously in super-Landau models can be 'exorcized'.

In order to simplify the calculations we will restrict ourselves here to the planar super-Landau models. We address the issue of the uniqueness, or otherwise, of the Hilbert space norm by adapting the methods of $\mathcal{PT}$-symmetric quantum theory (see [6] for a review) and 'bi-orthogonal systems' (see e.g. [7]). In that context one is given a Hamiltonian that fails to be hermitian with respect to a 'naive' norm and one considers whether it is possible to deform the norm in such a way that the Hamiltonian becomes hermitian. In our case, the starting Hamiltonian is hermitian but with respect to an indefinite norm and we need to modify the norm so that it becomes positive and is such that the Hamiltonian remains hermitian. The two problems are quite different but, in either case, the formalism is well-adapted to study the consequences for symmetries of a change of norm. Our conclusion will be that for the planar superflag there are *two* possible $ISU(1|1)$ invariant norms, one being

[1] A recent article [5] shows that even 'bosonic' ghosts need not lead to a violation of unitarity in quantum mechanics.



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the norm used in [1−3]. For $M<0$, it is the 'other' possible norm that is both positive and $ISU(1|1)$ invariant, but zero-norm states appear for $M=0$ and positivity for $M>0$ requires a 'dynamical' combination of both possible norms.

The issue of the Hilbert space norm was indirectly brought to our attention by a paper of Hasebe [8] on an alternative 'superplane' Landau model obtained as the planar limit of a spherical super-Landau model for a particle on the coset superspace $OSp(1|2)/\mathrm{U}(1)$ [9]. A feature of $OSp(1|2)/\mathrm{U}(1)$, which is also referred to as a 'supersphere' by many authors, is that the 'fermions' transform as an SU(2) doublet, which means that they must be complex because the doublet of SU(2) is pseudo-real rather than real. This feature carries over to the planar limit, so the 'superplane' of Hasebe is a superspace of real dimension (2|4) in contrast to the (2|2)-dimensional superplane of [3], but it can be interpreted as a superspace of 'pseudo-real' dimension (2|2) and it appears that the distinction is not relevant to the quantum theory. A further difference between [3] and [8] is that wave-functions were interpreted in [3] as superfields (functions of definite Grassmann parity), and this leads to a 'Hilbert' supervector space rather than to a standard vector space. In contrast, the coefficients in the $\zeta$-expansion of the wave-functions in [8] are all standard complex functions, and the norm for the Hilbert space they span is the positive-definite one. A remarkable feature of this choice is that the quantum theory can then be interpreted as a model of supersymmetric quantum mechanics (SQM); specifically, it has an unbroken $\mathcal{N}=2$ 'worldline' supersymmetry,[2] the SQM ground states forming the lowest Landau level [8].

The emergence of worldline supersymmetry is remarkable because it has no obvious connection to the 'internal' supersymmetry that underlies the model's construction, and a major purpose of this paper is to elucidate its origin. As we shall see, the transition from the indefinite norm to the positive-definite one leads to a change in the conjugation properties of 'fermionic' operators, which is effected via a 'shift' operator, and this leads to the Hamiltonian appearing as a central charge in the $ISU(1|1)$ algebra. In addition, the shift operator turns out to be the supersymmetry charge of the worldline supersymmetry algebra. A remarkable feature of the worldline supersymmetry generators is that they have a Sugawara-type realization in terms of the original $ISU(1|1)$ generators, although this feature is absent in a new 'natural' basis for which the symmetry algebra is manifestly the direct sum of the Lie superalgebra of $ISU(1|1)$ and the $\mathcal{N}=2$ worldline supersymmetry superalgebra.

The status of worldline supersymmetry is considerably clarified by consideration of the planar superflag Landau models. The additional anticommuting variable of these models was identified in [3] as a Nambu-Goldstone variable for the $ISU(1|1)$ supersymmetry. However, this variable is actually $ISU(1|1)$-inert in the 'natural' basis, and instead transforms inhomogeneously under worldline supersymmetry (at least for $M \leq 0$, which we assume for the purposes of this introduction); it is therefore a Nambu-Goldstone variable for a spontaneously broken $\mathcal{N}=2$ worldline supersymmetry! In the quantum theory, this new anticommuting variable becomes the complex Grassmann-odd coordinate of worldline su-

[2] Here we adopt the convention that $\mathcal{N}$ is the number of *real* supercharges, in contrast to [8, 9], and [10], where $\mathcal{N}$ is the number of *complex* supercharges.

perspace, and the wavefunction becomes a worldline superfield, with an expansion in terms of $ISU(1|1)$ superfields.

The equivalence of the $M{=}0$ planar superflag model to the superplane model was proved in [3] for the indefinite norm, but we show here that it remains true for the new, positive, norm. This equivalence means that the worldline supersymmetry that is spontaneously broken for $M{<}0$ is restored when $M{=}0$. Classically, this is because $\xi$ becomes a pure-gauge variable in the classical ground state when $M = 0$. Quantum mechanically, the worldline supersymmetry restoration occurs because half the ground states have zero norm when $M{=}0$, and the physical ground states (defined as equivalence classes of states modulo the addition of a zero-norm state) are annihilated by the worldline supersymmetry operators. In other words, supersymmetry is restored at $M{=}0$ by virtue of a discontinuity in the spectrum at $M{=}0$. This is rather different from the usual state of affairs for a family of SQM models in which the spectrum changes continuously with the parameter, so that supersymmetry can be broken at some values of the parameter only if the Witten index vanishes [10]. Here, it is not the energy eigenvalues that depend on the parameter but the norms of the states, and this allows a discontinuity in the spectrum because the norms of some ground states can go to zero.

## 2. Preliminaries

It is useful to discuss first some of the general structures to be encountered later in specific models. The quantum systems of interest possess inner products which, while naturally defined, are not positive definite. Therefore, let us assume that there exists a complete system of energy eigenvectors $|f_A\rangle$ for the Hamiltonian, $H$, which obey

$$\langle f_A|f_B\rangle = (-)^{g(A)}\,\delta_{AB}\,, \tag{2.1}$$

where $g\,(A)$ is the grading[3]

$$g\,(A) = \begin{cases} 0 & : \quad A = a \\ 1 & : \quad A = \alpha \end{cases} . \tag{2.2}$$

The subset of indices $A = a$ indicates positive norm states, while the subset $A = \alpha$ indicates negative norm states, for all eigenvectors. In fact (2.1) defines a system of linear functionals

$$\mathcal{F}_A\,(f_B) = (-)^{g(A)}\,\delta_{AB}\,, \tag{2.3}$$

which upon a trivial redefinition can be cast in the standard biorthogonal form [7] (see also [11 − 13]).

The operation of *naive* hermitian conjugation (†) will be taken with respect to the non-positive-definite inner product. For all the models of interest here, $H$ will be naively hermitian with respect to this inner product:

$$H = H^{\dagger}\,. \tag{2.4}$$

[3]This should not be confused with the grading associated to Grassmann parity, with anticommuting variables being Grassmann-odd.

To define an improved inner product, and obtain only positive norms, we introduce a 'metric operator' $G$ that acts on the eigenvectors $|f_A\rangle$ to give

$$G\,|f_A\rangle \equiv |Gf_A\rangle = (-)^{g(A)}\,|f_A\rangle\,, \qquad G = G^{\dagger}\,. \tag{2.5}$$

Thus, $H$ commutes with the metric, essentially by definition of the grading. The new inner product is then defined by the following formula

$$\langle\langle f_A|f_B\rangle\rangle \equiv \langle Gf_A|f_B\rangle = \delta_{AB}\,. \tag{2.6}$$

The 'improved' hermitian conjugate $\mathcal{O}^{\ddagger}$, with respect to $\langle\langle\cdots\rangle\rangle$, of any operator $\mathcal{O}$, is given by

$$\langle Gf_A\,|\mathcal{O}|\,f_B\rangle = \left\langle \mathcal{O}^{\dagger}Gf_A|f_B\right\rangle = \left\langle G\left(G^{-1}\mathcal{O}^{\dagger}G\right)f_A|f_B\right\rangle. \tag{2.7}$$

That is to say,

$$\langle\langle f_A\,|\mathcal{O}|\,f_B\rangle\rangle = \left\langle\left\langle \mathcal{O}^{\ddagger}f_A|f_B\right\rangle\right\rangle, \tag{2.8}$$

where

$$\mathcal{O}^{\ddagger} \equiv G^{-1}\mathcal{O}^{\dagger}G = \mathcal{O}^{\dagger} + S_{\mathcal{O}}\,. \tag{2.9}$$

Here we have introduced a "shift operator" for a given $\mathcal{O}$, as defined by

$$S_{\mathcal{O}} \equiv G^{-1}\left[\mathcal{O}^{\dagger}, G\right]. \tag{2.10}$$

Operators which do not commute with $G$ will have $\mathcal{O}^{\ddagger} \neq \mathcal{O}^{\dagger}$.

Note that $G = G^{\dagger}$ implies $\left(\mathcal{O}^{\ddagger}\right)^{\ddagger} = \mathcal{O}$, so that the new hermitian conjugation procedure closes in the familiar way. Correspondingly, the shift operators have the simple, useful conjugation property

$${S_{\mathcal{O}}}^{\ddagger} = -{S_{\mathcal{O}}}^{\dagger}\,. \tag{2.11}$$

As a consequence, the combination

$$\tilde{\mathcal{O}} \equiv \mathcal{O} + \frac{1}{2}S^{\dagger}_{\mathcal{O}} \tag{2.12}$$

has a conjugation with respect to the metric that coincides with its naive hermitian conjugate

$$\tilde{\mathcal{O}}^{\ddagger} = \tilde{\mathcal{O}}^{\dagger}\,. \tag{2.13}$$

We are going to extensively use all of these properties, as well as the following proposition **[Lemma]** Since $[G,H]=0$, the Hamiltonian $H$ is hermitian in both inner products, $H = H^{\dagger} = H^{\ddagger}$. Moreover, if the operator $\mathcal{O}$ is a constant of motion, then the corresponding shift operator is also a constant of motion. Indeed, from $[\mathcal{O},H]=0$ it follows that $\left[\mathcal{O}^{\dagger},H\right]=0$ *and* $\left[\mathcal{O}^{\ddagger},H\right]=0$. This is a signal that the algebra of operators which are in involution with the Hamiltonian may be larger than originally assumed: the system may have some 'hidden' symmetries.

## 3. Fermionic Landau model

The fermionic Landau model [8, 3] has the Lagrangian

$$L_f = \dot{\zeta}\dot{\bar{\zeta}} - i\kappa\left(\dot{\zeta}\bar{\zeta} + \dot{\bar{\zeta}}\zeta\right), \tag{3.1}$$

where $\kappa$ is a real positive constant, $\zeta(t)$ is a complex anticommuting variable with complex conjugate $\bar{\zeta}(t)$, and the overdot indicates its derivative with respect to the time parameter $t$. The equivalent phase space Lagrangian is

$$\tilde{L}_f = -i\dot{\zeta}\pi - i\dot{\bar{\zeta}}\bar{\pi} - H_f\,, \qquad H_f = (\bar{\pi} - \kappa\zeta)\left(\pi - \kappa\bar{\zeta}\right), \tag{3.2}$$

where $\pi$ ($\bar{\pi}$) is the momentum conjugate to $\zeta$ ($\bar{\zeta}$). To quantize, we make the replacements

$$\pi \to \partial_\zeta\,, \qquad \bar{\pi} \to \partial_{\bar{\zeta}}\,, \tag{3.3}$$

where the Grassmann-odd derivatives should be understood as left derivatives. With a standard operator ordering prescription, the quantum Hamiltonian is

$$H_f = \frac{1}{2}\left[\alpha, \alpha^\dagger\right] = -\alpha^\dagger\alpha - \kappa\,, \tag{3.4}$$

where

$$\alpha = \left(\partial_{\bar{\zeta}} - \kappa\zeta\right), \qquad \alpha^\dagger = \left(\partial_\zeta - \kappa\bar{\zeta}\right). \tag{3.5}$$

These operators satisfy the anticommutation relations

$$\left\{\alpha, \alpha^\dagger\right\} = -2\kappa\,. \tag{3.6}$$

The quantum Noether charges generating translations and phase rotation of the complex Grassmann plane parametrized by $\zeta$ are the differential operators

$$\Pi = \partial_\zeta + \kappa\bar{\zeta}\,, \qquad \Pi^\dagger = \partial_{\bar{\zeta}} + \kappa\zeta\,, \qquad F = \zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}}\,. \tag{3.7}$$

These span an 'internal' superalgebra for which the non-zero (anti)commutators are

$$\{\Pi, \Pi^\dagger\} = 2\kappa\,, \qquad [F, \Pi] = -\Pi\,, \qquad [F, \Pi^\dagger] = \Pi^\dagger\,. \tag{3.8}$$

It is straightforward to check that these generators commute with the Hamiltonian (3.4). Note that the Hamiltonian can be written as

$$H_f = \Pi^\dagger\Pi - 2\kappa F - \kappa\,, \tag{3.9}$$

which implies that it belongs to the enveloping algebra of the superalgebra defined by the relations (3.8).

A general wavefunction $\psi(\zeta, \bar{\zeta})$ is a function of $\zeta$ and $\bar{\zeta}$, which implies a total of four states. There are two ground states of energy $-\kappa$, with wavefunction $\psi^{(0)}$ annihilated by $\alpha$, and two excited states of energy $\kappa$, with wavefunction $\psi^{(1)}$ annihilated by $\alpha^\dagger$. These energy eigenfunctions take the form

$$\psi^{(0)} = e^{-\kappa\zeta\bar{\zeta}}\,\psi_0(\zeta)\,, \qquad \psi^{(1)} = e^{\kappa\zeta\bar{\zeta}}\,\psi_1\left(\bar{\zeta}\right), \tag{3.10}$$

for *analytic* function $\psi_0$ and *anti-analytic* function $\psi_1$:

$$\psi_0 = A_0 + \zeta\, B_0\,, \qquad \psi_1 = A_1 + \bar{\zeta}\, B_1\,. \tag{3.11}$$

The 2-vectors $(A_0\,, B_0)\,,\ (A_1\,, B_1)$ form two irreducible representations of the supertranslation group defined above. Its generators have the following realization on $\psi_0$: $\Pi_0 = \partial_\zeta\,, \Pi_0^\dagger = 2\kappa\zeta\,, F_0 = \zeta\partial_\zeta\,.$

Now we must choose an inner product. There are two obvious ways to proceed and each is instructive. We consider them in turn.

### 3.1 Superspace approach

One natural choice of inner product is

$$\langle\phi|\psi\rangle = \int d\zeta d\bar{\zeta}\ \overline{\phi\left(\zeta,\bar{\zeta}\right)}\psi\left(\zeta,\bar{\zeta}\right). \tag{3.12}$$

This has the property that $\alpha$ and $\alpha^\dagger$ are hermitian conjugates, when viewed as operators on wavefunctions, which guarantees the hermiticity of $H_f$. In turn, this guarantees the orthogonality of the energy eigenstates $\psi^{(0)}$ and $\psi^{(1)}$. However, the product (3.12) also implies a negative norm for excited states. Indeed, one finds that

$$\left\langle\psi^{(0)}|\psi^{(0)}\right\rangle = 2\kappa\bar{A}_0A_0 + \bar{B}_0B_0\,, \qquad \left\langle\psi^{(1)}|\psi^{(1)}\right\rangle = -2\kappa\bar{A}_1A_1 - \bar{B}_1B_1\,. \tag{3.13}$$

Therefore, with respect to the inner product (3.12) for which the operators $\alpha$ and $\alpha^\dagger$ are conjugate to each other and $H_f$ is manifestly hermitian, the norm is not positive definite.[4]

We can circumvent this difficulty by redefining the dual state vectors. Let us choose the 'metric' operator to be

$$G = -\kappa^{-1}H_f\,. \tag{3.14}$$

Note that $G = G^\dagger$, as required, and that

$$G\left(\psi^{(0)} + \psi^{(1)}\right) = \psi^{(0)} - \psi^{(1)}\,, \tag{3.15}$$

which implies that the improved inner product is positive definite. As $G$ commutes with the Hamiltonian in this example, there are *no* shifts introduced for the hermitian conjugates of any of the symmetry generators, and hence no change in the (anti)commutation relations (3.8). However, the hermitian conjugation properties of non-conserved operators can change. In particular, we have

$$\alpha^\ddagger = -\alpha^\dagger\,. \tag{3.16}$$

The operators $\alpha$ and $\alpha^\ddagger$ have the commutation relations of fermionic annihilation and creation operators,

$$\left\{\alpha, \alpha^\ddagger\right\} = 2\kappa\,, \tag{3.17}$$

[4]Note that this is true irrespective of whether the $A$ and $B$ coefficients are both ordinary complex numbers or complex supernumbers with Grassmann-odd products $AB$.

and the Hamiltonian is formally the same[5] as the hamiltonian of a fermionic harmonic oscillator:

$$H_f = \alpha^{\ddagger}\alpha - \kappa\,. \tag{3.18}$$

Note that the conjugation properties of the coordinates and momenta *are* altered:

$$\zeta^{\ddagger} = \frac{1}{\kappa}\partial_\zeta\,, \qquad \left(\bar{\zeta}\right)^{\ddagger} = \frac{1}{\kappa}\partial_{\bar{\zeta}}\,, \tag{3.19}$$

and correspondingly

$$(\partial_\zeta)^{\ddagger} = \kappa\zeta\,, \qquad \left(\partial_{\bar{\zeta}}\right)^{\ddagger} = \kappa\bar{\zeta}\,. \tag{3.20}$$

That is, under the new conjugation the momentum canonically conjugate to a coordinate is also the coordinate's hermitian conjugate!

### 3.2 Matrix approach

A general wave function can be written as

$$\psi\left(\zeta,\bar{\zeta}\right) = \mathcal{A} + \zeta\mathcal{B} + \bar{\zeta}\mathcal{C} + \bar{\zeta}\zeta\mathcal{D}\,, \tag{3.21}$$

for constant complex coefficients $\mathcal{A},\mathcal{B},\mathcal{C}$ and $\mathcal{D}$. In principle, these constants could be general super-numbers but we again suppose *either* that they are ordinary complex numbers, in which case the Hilbert space is $\mathbb{C}^4$, *or* that $\psi$ is a superfield (i.e. has definite Grassmann parity) in which case the 'Hilbert' space is the supervector space $\mathbb{C}^{(2|2)}$. In either case, the action of $H_f$ is given by

$$H_f\ \psi\left(\zeta,\bar{\zeta}\right) = -\mathcal{D} - \kappa\zeta\mathcal{B} + \kappa\bar{\zeta}\mathcal{C} - \kappa^2\bar{\zeta}\zeta\mathcal{A}\,. \tag{3.22}$$

Clearly, any procedure involving this model can be stated directly in terms of $4\times4$ (super)matrices. Let us associate to the superfield wavefunction $\psi$ the column (super)vector[6]

$$\vec{\psi} = \begin{pmatrix} \mathcal{A} \\ \mathcal{D} \\ \mathcal{B} \\ \mathcal{C} \end{pmatrix}. \tag{3.23}$$

Independently of the grading assigned to the coefficients $\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$, the differential operator $H_f$ is then equivalent to the (super)matrix

$$\mathcal{H} = \begin{pmatrix} & -1 & & \\ -\kappa^2 & & & \\ & & -\kappa & \\ & & & \kappa \end{pmatrix}. \tag{3.24}$$

[5]The difference is the doublet degeneracy of the two energy eigenstates, which is the fermionic version of the infinite degeneracy of the energy levels of the bosonic Landau model. This doublet degeneracy is related to the symmetry under supertranslations, with algebra (3.7), just as the degeneracy in the bosonic Landau model is related to the invariance under the 'magnetic translations' defined in (4.11).

[6]Note the non-alphabetic ordering.



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Because of its block-diagonal form, it is manifest that this may be viewed either as a matrix or as a supermatrix. In either case it is non-hermitian with respect to the usual positive definite inner product (for which $\mathcal{H}^\dagger = \overline{\mathcal{H}^{\rm T}}$), but it is hermitian with respect to the metric

$$\mathcal{G} = \begin{pmatrix} \kappa^2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{3.25}$$

in the sense that $\mathcal{G}\mathcal{H} = \mathcal{H}^\dagger \mathcal{G}$; i.e. it is 'quasi-hermitian' [14].

For any $\kappa \neq 0$, the matrix $\mathcal{H}$ can be diagonalized by a non-unitary similarity transformation, $\mathcal{H} = \mathcal{S}^{-1}\mathcal{H}_{\rm D}\mathcal{S}$, and the construction of a positive definite inner product in terms of the usual orthonormal basis of the transformed system is then straightforward. The inverse similarity transformation then leads from this orthonormal basis back to a bi-orthogonal system (see the classic text [11]) which corresponds to the previous polynomial basis and an appropriate set of dual polynomials. That is to say, in terms of the original basis underlying (3.23), a suitable metric is $\mathcal{G} = \mathcal{S}^\dagger\mathcal{S}$, where for example

$$\mathcal{S} = \begin{pmatrix} \kappa/\sqrt{2} & 1/\sqrt{2} & & \\ -\kappa/\sqrt{2} & 1/\sqrt{2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{3.26}$$

The matrix approach is thus equivalent to the superfield approach. However, it is convenient only for finite-dimensional matrices, so we revert to the superspace approach in what follows.

## 4. The superplane model

The Lagrangian of the superplane model [8, 3] is the sum $L = L_f + L_b$ of the Lagrangian (3.1) of the fermionic Landau model and the Lagrangian

$$L_b = |\dot z|^2 - i\kappa\left(\dot z \bar z - \dot{\bar z} z\right), \tag{4.1}$$

of Landau's original 'bosonic' model, where $2\kappa$ can now be identified as the (positive) value of a uniform magnetic field. The phase space Lagrangian is

$$\tilde L = \left(\dot z p - i\dot\zeta\pi\right) + c.c. \; - \; H_{\rm class}\,, \tag{4.2}$$

where

$$H_{\rm class} = |p + i\kappa\bar z|^2 + \left(\bar\pi - \kappa\zeta\right)\left(\pi - \kappa\bar\zeta\right). \tag{4.3}$$

For a standard operator ordering prescription, the corresponding quantum Hamiltonian operator is

$$H = \partial_{\bar\zeta}\partial_\zeta - \partial_z\partial_{\bar z} + \kappa\left(\bar z\partial_{\bar z} + \bar\zeta\partial_{\bar\zeta} - z\partial_z - \zeta\partial_\zeta\right) + \kappa^2\left(z\bar z + \zeta\bar\zeta\right). \tag{4.4}$$

Introducing the boson creation and annihilation operators

$$a = i\left(\partial_{\bar z} + \kappa z\right) , \qquad a^\dagger = i\left(\partial_z - \kappa \bar z\right) , \qquad \left[a, a^\dagger\right] = 2\kappa \,, \tag{4.5}$$

and recalling the definition (3.5) of the fermion creation and annihilation operators, we find that

$$H = a^\dagger a - \alpha^\dagger \alpha \,. \tag{4.6}$$

Note the cancellation of the zero point energies.

The ground state wavefunction $\psi^{(0)}$ for the lowest Landau level is annihilated by both $a$ and $\alpha$ and hence takes the form

$$\psi^{(0)} = e^{-\kappa K_2}\, \psi^{(0)}_{an}(z,\zeta) \,, \tag{4.7}$$

for *analytic* function $\psi^{(0)}_{an}$ . Here we introduced the notation (see [3])

$$K_2 = |z|^2 + \zeta\bar\zeta \,. \tag{4.8}$$

For each ground state, there are two excited states at the first Landau level, with wavefunctions given by the action of *either* $a^\dagger$ *or* $\alpha^\ddagger$ on the ground-state wavefunction. The wavefunctions at higher Landau levels (with the energy $E_N = 2\kappa N$) are obtained similarly and have the same degeneracy. Thus the $N$th level Hilbert space has a wavefunction of the form

$$\psi^{(N)} = \left(-ia^\dagger\right)^N e^{-\kappa K_2}\psi^{(N)}_+(z,\zeta) - N\left(-ia^\dagger\right)^{N-1}\alpha^\dagger e^{-\kappa K_2}\psi^{(N)}_-(z,\zeta) \,, \tag{4.9}$$

where $\psi_\pm(z,\zeta)$ are two analytic functions of $z$ and $\zeta$, and the factors of $i$ and $N$ are included for convenience of comparison with our later results. We may write these analytic wavefunctions as

$$\psi^{(N)}_\pm(z,\zeta) = A^{(N)}_\pm(z) + \zeta B^{(N)}_\pm(z) \,, \tag{4.10}$$

where the $A$ and $B$ coefficients are now analytic functions of $z$; a four-fold degeneracy of the excited states, relative to the bosonic Landau model, is now manifest.

The Hamiltonian commutes with the 'magnetic translation' operators

$$P = -i\left(\partial_z + \kappa\bar z\right) , \qquad P^\dagger = -i\left(\partial_{\bar z} - \kappa z\right) \tag{4.11}$$

and with the supermagnetic translation operators $(\Pi, \Pi^\dagger)$ defined in (3.7). The non-zero (anti)commu- tation relations of these supertranslation operators are

$$[P, P^\dagger] = 2\kappa \,, \qquad \{\Pi^\dagger, \Pi\} = 2\kappa \,. \tag{4.12}$$

The Hamiltonian also commutes with the operators:

$$Q = z\partial_\zeta - \bar\zeta\partial_{\bar z} \,, \qquad Q^\dagger = \bar z\partial_{\bar\zeta} + \zeta\partial_z \,, \tag{4.13}$$

and

$$C = z\partial_z + \zeta\partial_\zeta - \bar z\partial_{\bar z} - \bar\zeta\partial_{\bar\zeta} \,. \tag{4.14}$$

These operators span the algebra of SU(1|1), for which the only non-zero (anti)commutation relation is

$$\{Q, Q^\dagger\} = C\,. \tag{4.15}$$

Including the operators $P$, $P^\dagger, \Pi$ and $\Pi^\dagger$ leads to the semi-direct product superalgebra $ISU(1|1)$. In particular,

$$[Q, P] = i\Pi\,, \qquad \{Q^\dagger, \Pi\} = iP\,, \qquad [C, P] = -P\,, \qquad [C, \Pi] = -\Pi\,. \tag{4.16}$$

For the Hamiltonian (4.6) there exists a representation in terms of the $ISU(1|1)$ charges, analogous to (3.9):

$$H = P^\ddagger P + \Pi^\ddagger \Pi - 2\kappa C\,. \tag{4.17}$$

### 4.1 Norm and modified $ISU(1|1)$ algebra

The natural $ISU(1|1)$-invariant inner product is such that states at different levels are orthogonal and states within the same level have inner product

$$\langle \phi | \psi \rangle = \int d\mu\, \overline{\phi\left(z, \bar z; \zeta, \bar\zeta\right)}\, \psi\left(z, \bar z; \zeta, \bar\zeta\right), \tag{4.18}$$

where $d\mu$ is the $ISU(1|1)$-invariant superspace measure

$$d\mu = dz d\bar z d\zeta d\bar\zeta\,. \tag{4.19}$$

As in the purely fermionic case, and for the same reason, this leads to negative norm states. Specifically, one finds that

$$\langle \psi^{(N)} | \psi^{(N)} \rangle \;=\; (2\kappa)^N N! \left[ -N \left|\left| \psi_-^{(N)} \right|\right|^2 + \left|\left| \psi_+^{(N)} \right|\right|^2 \right], \tag{4.20}$$

where we have defined

$$||\phi_{an}||^2 \equiv \int d\mu\; e^{-2\kappa K_2}\; \overline{\phi_{an}}\; \phi_{an} \tag{4.21}$$

for any *analytic* function, or superfield, $\phi_{an}(z, \zeta)$. A computation shows that[7]

$$\left|\left| \psi_\pm^{(N)} \right|\right|^2 = \int dz d\bar z\, e^{-2\kappa |z|^2} \left( 2\kappa \overline{A_\pm^{(N)}(z)} A_\pm^{(N)}(z) + \overline{B_\pm^{(N)}(z)} B_\pm^{(N)}(z) \right), \tag{4.22}$$

so the minus sign in (4.20) implies an indefinite norm. This problem is circumvented exactly as before, and with the same metric operator $G = -\kappa^{-1} H_f$, which we may write as

$$G = \frac{1}{\kappa} \left[ \partial_\zeta \partial_{\bar\zeta} + \kappa^2 \bar\zeta \zeta + \kappa \left( \zeta \partial_\zeta - \bar\zeta \partial_{\bar\zeta} \right) \right]. \tag{4.23}$$

It is evident that $G$ commutes with $H$. It is also easy to verify that $G$ commutes with the operators $a$ and $a^\dagger$, but *not* with $\alpha$ and $\alpha^\dagger$, and this leads to the modified hermitian conjugates

$$\alpha^\ddagger = -\alpha^\dagger\,, \tag{4.24}$$

[7]Recall that either the $A$ or the $B$ coefficient function will be Grassmann odd if the wavefunction is a superfield.



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as in the fermionic Landau model. The Hamiltonian may now be written in the manifestly positive form

$$H = a^\dagger a + \alpha^\ddagger \alpha\,. \tag{4.25}$$

The metric operator $G$ commutes with all the bosonic symmetry generators of $ISU(1|1)$, and the fermionic generators $\Pi$ and $\Pi^\dagger$, which therefore all have unchanged hermitian conjugates. However $G$ does *not* commute with $Q$, and this leads to the modified hermitian conjugate

$$Q^\ddagger = Q^\dagger - \frac{i}{\kappa} S\,, \tag{4.26}$$

where the *shift* operator is

$$S = i\left(\partial_z \partial_{\bar\zeta} + \kappa^2 \bar z \zeta - \kappa \bar z \partial_{\bar\zeta} - \kappa \zeta \partial_z\right). \tag{4.27}$$

As explained in section 2, it is convenient to introduce the new operator

$$\tilde Q = Q - \frac{i}{2\kappa} S^\ddagger\,, \tag{4.28}$$

since this operator commutes with $G$ and therefore has the property that $\tilde Q^\ddagger = \tilde Q^\dagger$. We now have

$$\left\{\tilde Q, \tilde Q^\dagger\right\} = \tilde C\,, \tag{4.29}$$

where

$$\tilde C = C + \frac{1}{2\kappa} H\,. \tag{4.30}$$

We now have two commuting symmetries, one an $ISU(1|1)$ symmetry with the modified charges $(P, \Pi, \tilde Q; P^\dagger, \Pi^\dagger, \tilde Q^\dagger; \tilde C)$, and the other a worldline supersymmetry algebra with charges $(S, S^\ddagger; H)$. The generator $\tilde C$ differs from the original $C$ by the term proportional to $H$, which commutes with all symmetry generators and so can be thought of as a central charge. Thus the new $ISU(1|1)$ algebra can be interpreted as a central extension of the original $ISU(1|1)$ algebra.

### 4.2 Worldline supersymmetry

As explained in section 2, the hermiticity of the Hamiltonian with respect to both the original and the modified norm implies that both $S$ and $S^\ddagger$ are constants of motion. These operators can be written as

$$S = a^\dagger \alpha\,, \qquad S^\ddagger = a \alpha^\ddagger\,, \tag{4.31}$$

and they have the anticommutation relation

$$\{S, S^\ddagger\} = 2\kappa H\,, \qquad \{S, S\} = 0 = \{S^\ddagger, S^\ddagger\}\,, \tag{4.32}$$

which is an $\mathcal{N}$=2 worldline supersymmetry algebra. Note also that

$$\{S, \tilde Q\} = 0\,, \qquad \{S, \tilde Q^\dagger\} = 0\,. \tag{4.33}$$

The worldline supersymmetry is unbroken because the ground state is annihilated by both $S$ and $S^\ddagger$. The ground state is a singlet of the $\mathcal{N}$=2 worldline supersymmetry, but still forms a non-trivial multiplet of $ISU(1|1)$, which explains its doublet degeneracy. All higher $N$ states form non-trivial multiplets of $\mathcal{N}$=2 worldline supersymmetry consisting of two irreducible $ISU(1|1)$ multiplets. This implies the four-fold degeneracy of these states.[8]

Classically, the supersymmetry charges generate transformations of the phase-space variables. After elimination of the momentum variables one finds that the infinitesimal transformation generated by $\epsilon S + \bar{\epsilon} S^{\ddagger}$, for complex anticommuting parameter $\epsilon$, is

$$\delta z = \epsilon \dot{\zeta} \,, \qquad \delta \zeta = -\dot{z}\bar{\epsilon} \,. \tag{4.34}$$

It is readily verified that the configuration space Lagrangian is invariant under these transformations, and that their algebra closes, on-shell, to the worldline supersymmetry algebra. This classical supersymmetry is unbroken by the classical ground state solutions, for which both $z$ and $\zeta$ are constant, as expected from the fact that worldline supersymmetry is unbroken quantum mechanically. The worldline supersymmetry is quite remarkable, taking into account the unconventional form of the above transformations; conventionally, $z$ would vary into some fermionic field of the proper dimension, not into its time derivative. This unconventional form means that the commutator on $z$ and $\zeta$ involves $\ddot{z}$ and $\ddot{\zeta}$ rather than $\dot{z}$ and $\dot{\zeta}$; but $\ddot{z}$ and $\dot{z}$ are related by the equations of motion, as are $\ddot{\zeta}$ and $\dot{\zeta}$, this being a characteristic feature of Landau models. For this reason, the on-shell closure of (4.34) involves $\kappa$, so the $\mathcal{N}$=2 supersymmetry is made possible by the WZ terms with non-zero coefficient $\kappa$.

Although worldline supersymmetry has emerged as an 'accidental' symmetry in the sense that it played no role in the construction of the model, there is another sense in which it is 'almost' built into the construction. This follows from the observation that the worldline supersymmetry generators belong to the enveloping algebra of $ISU(1|1)$, as is shown by the Sugawara-type representation

$$S = 2i\kappa Q^{\ddagger} + P\Pi^{\ddagger} \,, \qquad S^{\ddagger} = -2i\kappa Q + P^{\ddagger}\Pi \tag{4.35}$$

and (4.17). The anticommutation relations of (4.32) are now a direct consequence of the $ISU(1|1)$ (anti)commutation relations of (4.12), (4.15), (4.16).

It may appear from this result that worldline supersymmetry is an automatic consequence of $ISU(1|1)$ symmetry, but this is not quite true. Suppose that we try to similarly define supercharges $\tilde{S}$ and $\tilde{S}^{\ddagger}$ in terms of the modified $ISU(1|1)$ generators. We then have

$$\tilde{S} = 2i\kappa \tilde{Q}^{\ddagger} + P\Pi^{\ddagger} \,, \qquad \tilde{S}^{\ddagger} = -2i\kappa \tilde{Q} + P^{\ddagger}\Pi \tag{4.36}$$

and

$$\tilde{H} = P^{\ddagger}P + \Pi^{\ddagger}\Pi - 2\kappa \tilde{C} \,. \tag{4.37}$$

However, *these charges are identically zero*, as a consequence of the further Sugawara-type relations[9]

$$\tilde{Q} = -\frac{i}{2\kappa}\, P^{\dagger}\Pi \,, \qquad \tilde{Q}^{\dagger} = \frac{i}{2\kappa}\, P\Pi^{\dagger} \,, \tag{4.38}$$

[8] Of course, these degeneracies should be understood as relative to the bosonic Landau model.

[9] These relations show that the modified $ISU(1|1)$ supersymmetry belongs to the enveloping algebra of the superplane translation algebra.

and

$$\tilde{C} = \frac{1}{2\kappa}\left[P^{\ddagger}P + \Pi^{\ddagger}\Pi\right]. \tag{4.39}$$

Thus, worldline supersymmetry is not an *automatic* consequence of $ISU(1|1)$ invariance, and this was the reason for the qualification 'almost'. In fact, it should be obvious that the Sugawara construction cannot yield anything new in a 'natural' basis for the charges which makes manifest that the symmetry group is the direct product of $ISU(1|1)$ and worldline supersymmetry. So the apparently miraculous construction of the worldline supersymmetry algebra from the $ISU(1|1)$ algebra is really just a consequence of the fact that we did not initially obtain the generators in their natural basis.

In order to better understand the origin of worldline supersymmetry, we now turn to the planar superflag models.

## 5. The planar superflag model

The superflag Landau model [2] describes a charged particle on the coset superspace SU(2|1)/[U(1) × U(1)]. One of the two Wess-Zumino (WZ) terms associated with the U(1) × U(1) group is the Lorentz coupling to a uniform magnetic field of strength $2\kappa$, where $\kappa$ can be identified as the constant already introduced in the previous sections. The second WZ term is a purely 'fermionic' one with constant coefficient $M$. The details may be found in [2]; here we are concerned with the planar limit, in which one finds the following $ISU(1|1)$-invariant Lagrangian [3]:

$$\begin{aligned} L = &\left(1+\bar{\xi}\xi\right)|\dot{z}|^2 + \left(\bar{\xi}\dot{\bar{z}}\dot{\zeta} - \xi\dot{z}\dot{\bar{\zeta}}\right) + \bar{\xi}\xi\dot{\zeta}\dot{\bar{\zeta}} \\ &- i\kappa\left(\dot{z}\bar{z} - \dot{\bar{z}}z + \dot{\zeta}\bar{\zeta} + \dot{\bar{\zeta}}\zeta\right) + iM\left(\bar{\xi}\dot{\xi} + \xi\dot{\bar{\xi}}\right). \end{aligned} \tag{5.1}$$

Notice that $\xi$ is auxiliary when $M$=0; its elimination returns us to the superplane Lagrangian[10] so we now have a one-parameter deformation of the superplane Landau model that both preserves the $ISU(1|1)$ symmetry and retains the property that the bosonic truncation yields Landau's original model. The new variable $\xi$ was interpreted in [3] as a Nambu-Goldstone variable associated with the spontaneous breaking of the $ISU(1|1)$ 'supersymmetry', generated by the Noether charge $Q$. However, we shall see (at least for $M$<0) that its interpretation in the quantum theory with positive norm is as a Nambu-Goldstone variable for the spontaneous breakdown of an $\mathcal{N}$=2 worldline supersymmetry.

It will be instructive to consider the classical theory before turning to the quantum theory. Introducing the momentum variables $(p,\pi)$ conjugate to $(z,\zeta)$, we can express the Lagrangian in the alternative form[11]

$$L = \left\{\left[\dot{z}p - i\dot{\zeta}\pi - iM\dot{\xi}\bar{\xi}\right] + \lambda\varphi\right\} + c.c \ - H_{\rm class}\,, \tag{5.2}$$

[10]Assuming that $\dot{z} \neq 0$; this is a subtlety dealt with in [3], where the quantum equivalence of the $M$=0 planar superflag model to the superplane model was established.

[11]This is essentially eq. (3.7) of [3] after using the phase space constraint $\varphi_\xi \approx 0$ to eliminate the momentum variable $\chi$ conjugate to $\xi$, but with $\tilde{p}$ of that reference written here as $p$.

where

$$H_{\rm class} = \left(1 - \bar\xi\xi\right) |p + i\kappa\bar z|^2 \tag{5.3}$$

is the classical Hamiltonian, and $\lambda$ is a Lagrange multiplier for the constraint $\varphi \approx 0$, where

$$\varphi = \pi - \kappa\bar\zeta + i\bar\xi \left(p + i\kappa\bar z\right). \tag{5.4}$$

If this constraint is used to eliminate $\pi$, we get a Lagrangian in terms of the complex variables $(z, \zeta, \xi, p)$, for which the Euler-Lagrange equations are equivalent to

$$\begin{aligned} \dot z &= \left(1 + \frac{i}{2\kappa}\bar\xi\dot\xi\right)\left(\bar p - i\kappa z\right), \qquad \dot p = i\kappa\dot{\bar z}, \\ \dot\zeta &= -\left[\xi + \frac{i}{2\kappa}\dot\xi\left(1 - \bar\xi\xi\right)\right]\left(\bar p - i\kappa z\right), \end{aligned} \tag{5.5}$$

and

$$\left[H_{\rm class} - 4\kappa M\right]\dot\xi = 0. \tag{5.6}$$

This last equation shows that $\dot\xi = 0$, except when the energy equals $4\kappa M$. This is never the case when $M<0$, so the equations of motion for $M<0$ are equivalent to

$$\dot z = \left(\bar p - i\kappa z\right), \qquad \dot p = i\kappa\dot{\bar z}, \tag{5.7}$$

and

$$\dot\zeta = -\xi\dot z, \qquad \dot\xi = 0. \tag{5.8}$$

These equations imply the superplane Landau model equations of motion

$$\ddot z = -2i\kappa\dot z, \qquad \ddot\zeta = -2i\kappa\dot\zeta. \tag{5.9}$$

However, whereas the initial conditions for the superplane model are the values at a given time of $(z, \dot z, \zeta, \dot\zeta)$, the initial conditions for the equations of the $M<0$ planar superflag model are the values of $(z, \dot z, \zeta, \xi)$. These are equivalent as long as $\dot z \neq 0$, because then $\xi = -\dot\zeta/\dot z$ but they are inequivalent at $\dot z = 0$. Specifically, $\dot z = 0$ implies $\dot\zeta = 0$ for the planar superflag model, but $\xi$ is then undetermined. This implies that $\xi$ is an independent, albeit constant, variable in a classical ground state, for which $H_{\rm class} = 0$. This is also true for $M = 0$ (where $\xi$ is auxiliary for $\dot z \neq 0$) but in this case (i) $\xi$ need not be constant because (5.6) is an identity when $H_{\rm class} \sim |\dot z|^2 = 0$, and (ii) $\xi(t)$ can be 'gauged away' by a fermionic gauge invariance, as shown in [3] (where it was also shown that a similar gauge invariance arises when $2M$ is any non-negative integer). The significance of these facts will become apparent when we discuss worldline supersymmetry, but let us stress here the independence of the classical physics on $M$ as long as $M<0$. As we should expect, we will find that the same is true of the quantum theory.

### 5.1 Quantum theory

The quantization of the planar superflag model is complicated by the phase-space constraint. In particular, the classical Hamiltonian $H_{\rm class}$ does not have weakly vanishing Poisson brackets with the constraint function $\varphi$ and its complex conjugate $\bar\varphi$. This problem was dealt with in [3] by a change of variables but it was noted that an alternative approach would be to consider the modified Hamiltonian[12]

$$H'_{\rm class} = \left(1+\bar\xi\xi\right)\left|p + i\kappa\bar z + i\xi\left(\pi - \kappa\bar\zeta\right)\right|^2 , \tag{5.10}$$

which is weakly equal to $H_{\rm class}$ and has weakly vanishing Poisson brackets with both $\varphi$ and $\bar\varphi$. This alternative approach is much more convenient for present purposes. The results obtained in this way are of course equivalent to those of [3], but the wavefunctions are now functions of $(z,\zeta,\xi)$. Following [3], we define

$$K_1 = 1 + \bar\xi\xi \tag{5.11}$$

and introduce the 'shifted' $z$ variable

$$z_{sh} = z + \bar\xi\zeta\,, \qquad \bar z_{sh} = \bar z - \xi\bar\zeta\,. \tag{5.12}$$

We may now quantize without constraint provided that we restrict to 'physical' wavefunctions, which take the form

$$\Psi = K_1^M e^{-\kappa K_2}\Psi_{ch}\left(z,\bar z_{sh},\zeta,\xi\right), \tag{5.13}$$

where $\Psi_{ch}$ is a 'chiral' wavefunction that depends on $\bar\zeta$ only through $\bar z_{sh}$, and $K_2$ was defined in (4.8); we refer to [3] for details. The Hamiltonian operator acting on these wavefunctions can be written as

$$H = \hat a^\dagger \hat a\,, \tag{5.14}$$

where the 'non-linear' annihilation and creation operators

$$\hat a = i\sqrt{K_1}\left(\partial_{\bar z} + \kappa\, z_{sh} - \bar\xi\partial_{\bar\zeta}\right), \qquad \hat a^\dagger = i\sqrt{K_1}\left(\partial_z - \kappa\,\bar z_{sh} - \xi\partial_\zeta\right), \tag{5.15}$$

have the same commutation relation as for the bosonic Landau model:

$$\left[\hat a, \hat a^\dagger\right] = 2\kappa\,. \tag{5.16}$$

In writing the Hamiltonian operator as (5.14) we are resolving the operator ordering ambiguity by a 'normal ordering' prescription that differs from the 'harmonic oscillator' prescription that we used for the bosonic Landau model (and in [3]). As a consequence, $H$ has eigenvalues $2\kappa N$, where $N$ is a non-negative integer, exactly as for the superplane Landau model. In the physical energy eigenfunctions at level $N$ the chiral wavefunction is expressed through an *analytic* function of $(z,\zeta,\xi)$ as

$$\Psi^{(N)}_{ch} = \tilde\nabla_z^N \Psi^{(N)}_{an}\left(z,\zeta,\xi\right), \qquad \tilde\nabla_z = \partial_z - 2\kappa\bar z_{sh} - \xi\partial_\zeta\,. \tag{5.17}$$

[12]This is eq. (3.22) of [3] but with $\tilde p$ now written as $p$.



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It is useful to note that any physical operator $\mathcal{O}$ is defined by its action on the energy eigenfunctions $\Psi^{(N)}$, and if it commutes with the Hamiltonian then this action is determined by an associated 'short' operator $\mathcal{O}_{an}$ acting on the associated analytic wavefunctions $\Psi_{an}^{(N)}$:

$$\mathcal{O}\,\Psi^{(N)} = K_1^M\, e^{-\kappa K_2}\tilde{\nabla}_z^N\, \mathcal{O}_{an}\Psi_{an}^{(N)}\,. \tag{5.18}$$

In particular, the short form of the Hamiltonian operator is

$$H_{an} = 2\kappa N_{an}\,, \tag{5.19}$$

where $N_{an}$ is the 'short' level number operator defined by

$$N_{an}\Psi_{an}^{(N)} = N\,\Psi_{an}^{(N)}\,. \tag{5.20}$$

As the operators generating the $ISU(1|1)$ symmetry commute with the Hamiltonian, they too may be represented by their short forms, which are

$$\begin{aligned}
P_{an} &= -i\partial_z\,, & P_{an}^\dagger &= 2i\kappa z\,,\\
\Pi_{an} &= \partial_\zeta\,, & \Pi_{an}^\dagger &= 2\kappa\zeta\,,\\
Q_{an} &= z\partial_\zeta - \partial_\xi\,, & Q_{an}^\dagger &= \zeta\partial_z + (N_{an}-2M)\,\xi\,,\\
C_{an} &= \zeta\partial_\zeta + z\partial_z + 2M - N_{an}\,.
\end{aligned} \tag{5.21}$$

One may verify that the associated operators $(P, P^\dagger; \Pi, \Pi^\dagger; Q, Q^\dagger; C)$, defined via (5.18), satisfy the $ISU(1|1)$ (anti)commutation relations (4.12), (4.15) and (4.16), and that $(P^\dagger, \Pi^\dagger, Q^\dagger)$ are the hermitian conjugates of $(P, \Pi, Q)$, with respect to the $ISU(1|1)$-invariant inner product:

$$\langle\Phi\big|\Psi\rangle = \int d\mu \int d\xi d\bar{\xi}\;\overline{\Phi}\,\Psi \;\; = \int d\mu\, e^{-2\kappa K_2}\int d\xi d\bar{\xi}\;K_1^{2M}\,\overline{\Phi_{ch}}\,\Psi_{ch}\,, \tag{5.22}$$

where $d\mu = dzd\bar{z}d\zeta d\bar{\zeta}$ is the measure of (4.19). More generally, for any 'physical' operator $\mathcal{O}$ (i.e. one that acts on 'physical' wavefunctions), we define $\mathcal{O}^\dagger$ to be its hermitian conjugate with respect to this inner product.

When acting on physical states, the Hamiltonian can be written as

$$H = P^\ddagger P + \Pi^\ddagger\Pi - 2\kappa C + 4\kappa M\,, \tag{5.23}$$

which is analogous to (4.17), with which it coincides for $M = 0$. It follows that H is hermitian with respect to the above inner product, and the hermiticity of H implies that energy eigenfunctions in different Landau levels are orthogonal. The 'superflag' norm of an energy eigenfunction within the $N$th level is given by

$$\Big|\Big|\Psi^{(N)}\Big|\Big|_{sf}^2 \equiv \langle\Psi^{(N)}\big|\Psi^{(N)}\rangle = \int d\mu\, e^{-2\kappa K_2}\int d\xi d\bar{\xi}\;K_1^{2M}\,\Big|\tilde{\nabla}_z^N\Psi_{an}^{(N)}\Big|^2. \tag{5.24}$$

We may write

$$\Psi_{an}^{(N)} = \psi_-^{(N)}(z,\zeta) + \xi\,\psi_+^{(N)}(z,\zeta) \tag{5.25}$$

and

$$\tilde{\nabla}_z = \tilde{D}_z + \xi \left(2\kappa\bar{\zeta} - \partial_\zeta\right), \qquad \tilde{D}_z = \partial_z - 2\kappa\bar{z}\,, \tag{5.26}$$

to get

$$\tilde{\nabla}^N \Psi^{(N)}_{an} = \tilde{D}^N_z \psi^{(N)}_- + \xi \left[\tilde{D}^N_z \psi^{(N)}_+ + N\tilde{D}^{N-1}_z \left(2\kappa\bar{\zeta} - \partial_\zeta\right)\psi^{(N)}_-\right]. \tag{5.27}$$

Performing the Berezin integration over $\xi$ and $\bar{\xi}$ in (5.24) then gives

$$\left|\left|\Psi^{(N)}\right|\right|^2_{sf} = \int d\mu e^{-2\kappa K_2}\left\{2M\left|\tilde{D}^N_z\psi^{(N)}_-\right|^2 + \left|\tilde{D}^{(N)}_z\psi^{(N)}_+ + N\left(2\kappa\bar{\zeta}-\partial_\zeta\right)\tilde{D}^{N-1}_z\psi^{(N)}_-\right|^2\right\}. \tag{5.28}$$

The cross term in the expansion of the final term in this expression is zero, as can be proved by integration by parts of the term with $\partial_\zeta$. We thus have

$$\left|\left|\Psi^{(N)}\right|\right|^2_{sf} = \int d\mu e^{-2\kappa K_2}\left\{2M\left|\tilde{D}^N_z\psi^{(N)}_-\right|^2 + \left|\tilde{D}^{(N)}_z\psi^{(N)}_+\right|^2 + N^2\left|(2\kappa\bar{\zeta}-\partial_\zeta)\tilde{D}^{N-1}_z\psi^{(N)}_-\right|^2\right\}. \tag{5.29}$$

One may further show by integration by parts that

$$\int d\mu\, e^{-2\kappa K_2}\left|\left(2\kappa\bar{\zeta} - \partial_\zeta\right)\tilde{D}^{N-1}_z\psi^{(N)}_-\right|^2 = -2\kappa\int d\mu\, e^{-2\kappa K_2}\left|\tilde{D}^{N-1}_z\psi^{(N)}_-\right|^2 \tag{5.30}$$

and also that

$$\int d\mu\, e^{-2\kappa K_2}\left|\tilde{D}^J_z\psi^{(N)}_\pm\right|^2 = (2\kappa)^J J!\int d\mu\, e^{-2\kappa K_2}\left|\psi^{(N)}_\pm\right|^2, \tag{5.31}$$

for any integer $J$. We thus find that

$$\left|\left|\Psi^{(N)}\right|\right|^2_{sf} = (2\kappa)^N N!\left[(2M-N)\left|\left|\psi^{(N)}_-\right|\right|^2 + \left|\left|\psi^{(N)}_+\right|\right|^2\right], \tag{5.32}$$

where the norm on the right hand side is the 'analytic-function norm' defined in (4.20) and given explicitly for the $\psi_\pm$ analytic functions in (4.21), (4.22). This is the result of [3]. With this norm, there are ghosts in the levels with $N>2M$, and zero-norm states in the level with $N=2M$ whenever $2M$ is a non-negative integer. Note the agreement with (4.20) for $M=0$, which is a consequence of the equivalence of the $M=0$ model with the superplane model for the 'naive' superspace norm.

### 5.2 Positive inner product

The inner product (5.22) is not unique but if we wish to preserve the $ISU(1|1)$ invariance then any planar superflag metric operator $G_{sf}$ yielding a new inner product must be a function only of $\xi$ and $\partial_\xi$. If we also require that $G_{sf}$ have even Grassmann parity and is such that $G^2_{sf}=1$, then there are only two possibilities for its 'short' form: either $G_{an}=1$, which implies $G_{sf}=1$ (as in [3] and assumed so far), or

$$G_{an} = [\xi, \partial_\xi] = -1 + 2\xi\partial_\xi\,. \tag{5.33}$$

One may verify that the corresponding operator $G_{sf}$ has all the properties required of a metric operator. Observing that

$$G_{an}\Psi^{(N)}_{an} = -\psi^{(N)}_- + \xi\,\psi^{(N)}_+\,, \tag{5.34}$$

we deduce that the new norm of $\Psi^{(N)}$ is

$$\langle\langle \Psi^{(N)} \big| \Psi^{(N)} \rangle\rangle \equiv \langle \Psi^{(N)} \big| G_{sf}\, \Psi^{(N)} \rangle \propto (N-2M) \left|\left| \psi_-^{(N)} \right|\right|^2 + \left|\left| \psi_+^{(N)} \right|\right|^2 . \tag{5.35}$$

All states now have positive norm when $M{<}0$. This remains true for $M{=}0$ except that half of the $N{=}0$ states, namely those comprised by $\Psi_-^{(0)}$, have zero norm. When there are zero-norm states, the vector (super)space of physical states is the quotient of the space of all states by the subspace of zero-norm states, which means that any state of zero-norm corresponds to the zero-vector of the physical space. Thus, zero-norm states do not contribute to the physical spectrum. Taking this into account, it follows that the $M{=}0$ planar superflag model has precisely the same spectrum, including degeneracies, as the superplane model, and is therefore equivalent to it.

In view of this equivalence, our choice of superflag metric operator $G_{sf}$ should imply, for $M{=}0$, the superplane metric operator $G$ of (4.23). To verify this, we note that the superflag wavefunction $\Psi^{(N)}$ has the $\xi$-expansion

$$\Psi^{(N)} = \Big(-ia^\dagger\Big)^N e^{-\kappa K_2} \psi_-^{(N)} + \xi \left[ \Big(-ia^\dagger\Big)^N e^{-\kappa K_2} \psi_+^{(N)} - N\Big(-ia^\dagger\Big)^{N-1} \alpha^\dagger e^{-\kappa K_2} \psi_-^{(N)} \right], \tag{5.36}$$

where $a^\dagger$ and $\alpha^\dagger$ are the superplane creation operators introduced in (3.5) and (4.5). Noting that

$$\partial_\xi \Psi^{(N)} = \psi^{(N)} , \tag{5.37}$$

where $\psi^{(N)}$ is precisely the superplane energy eigenfunction of (4.9), we see that

$$\int d\mu \int d\xi d\bar{\xi}\; \overline{\Psi^{(N)}}\, \Psi^{(N)} = \int d\mu\; \overline{\psi^{(N)}}\, \psi^{(N)} , \tag{5.38}$$

and hence the 'naive' $M = 0$ superflag norm coincides with the 'naive' superplane norm, as expected. We now observe that

$$\partial_\xi \left( G_{sf} \Psi^{(N)} \right) = G\, \psi^{(N)} , \tag{5.39}$$

from which it follows that the modified planar superflag norm implies the modified norm introduced earlier for the superplane model.

When $M{>}0$ there are negative-norm states for all $N{<}2M$, in particular for $N{=}0$, but one can revert to the 'naive' norm for these levels, thus ensuring that all states have a positive-definite (or zero) norm for any value of $M$. Note that the two norms coincide when $N{=}2M$, which can happen only when $2M$ is a non-negative integer, and in this case there are zero-norm states. The $M{=}0$ case discussed above is just a special case of this phenomenon. Taking into account the possibility of zero-norm states, we see that the spectrum is the same for all $M$, with the same degeneracy at each Landau level, except when $2M$ is a non-negative integer. Every such non-negative integer yields a different spectrum because half of the states in the $N{=}2M$ level have zero norm. In what follows we assume that $M{<}0$ so that the metric operator $G_{sf}$ is given by (5.33); the modification required for the $N{<}2M$ states when $M{>}0$ will be obvious since the 'naive' norm then applies.

The only $ISU(1|1)$ generators that fail to commute with $G_{sf}$ are $Q$ and $Q^\dagger$:

$$[G_{an}, Q_{an}] = 2\partial_\xi\,, \qquad \left[G_{an}, Q^\dagger_{an}\right] = 2\xi\,(N_{an} - 2M)\,. \tag{5.40}$$

This means that all hermitian conjugates are unmodified except for those of $Q$ and $Q^\dagger$. Following the general procedure, we have

$$Q^\ddagger_{an} = G_{an}\, Q^\dagger_{an}\, G_{an} = \left(Q^\dagger_{an} - \frac{i}{\kappa} S_{an}\right), \tag{5.41}$$

$$(Q^\dagger_{an})^\ddagger = (Q^\ddagger_{an})^\dagger = G_{an}\, Q_{sh}\, G = \left(Q_{an} + \frac{i}{\kappa} S^\dagger_{an}\right), \tag{5.42}$$

whence, using (5.40), the shift operators $S$ and $S^\dagger$ are found to be

$$S_{an} = 2i\kappa\xi\,(2M - N_{an})\,, \qquad S^\dagger_{an} = -2i\kappa\partial_\xi\,. \tag{5.43}$$

The shift operators do not commute with $G$, since

$$[G_{an}, S_{an}] = 2S_{an}\,, \qquad [G_{an}, S^\dagger_{an}] = -2S^\dagger_{an}\,, \tag{5.44}$$

and hence $S^\dagger$ is no longer the hermitian conjugate of $S$. In fact, its hermitian conjugate is $S^\ddagger = -S^\dagger$, and hence

$$\{S_{an}, S^\ddagger_{an}\} = 4\kappa^2\,(N_{an} - 2M)\,. \tag{5.45}$$

Again following the general procedure, we define 'improved' $ISU(1|1)$ supersymmetry generators

$$\tilde{Q}_{an} = Q_{an} + \frac{i}{2\kappa} S^\dagger_{an} = z\partial_\zeta\,. \tag{5.46}$$

As this operator commutes with $G_{an}$, we have

$$\tilde{Q}^\ddagger_{an} = \tilde{Q}^\dagger_{an} = Q^\dagger_{an} - \frac{i}{2\kappa} S_{an} = \zeta\partial_z\,. \tag{5.47}$$

If we now define the new U(1) generator

$$\tilde{C}_{an} = C_{an} + (N_{an} - 2M)\,, \tag{5.48}$$

which yields precisely the same redefinition as in (4.30), then the new $ISU(1|1)$ generators are

$$\begin{aligned} P_{an} &= -i\partial_z\,, & P^\ddagger_{an} &= 2i\kappa z\,,\\ \Pi_{an} &= \partial_\zeta\,, & \Pi^\ddagger_{an} &= 2\kappa\zeta\,,\\ \tilde{Q}_{an} &= z\partial_\zeta\,, & \tilde{Q}^\ddagger_{an} &= \zeta\partial_z\,,\\ \tilde{C}_{an} &= z\partial_z + \zeta\partial_\zeta. \end{aligned} \tag{5.49}$$

One may verify that these operators obey the (anti)commutation relations of $ISU(1|1)$.

As the 'analytic' $ISU(1|1)$ generators now act on functions of $(z, \zeta)$ alone, they evidently (anti)commute with $S_{an}$ and $S^\ddagger_{an}$, which act on functions of $\xi$ alone. As a consequence, the variable $\xi$ can no longer be interpreted as a Nambu-Goldstone variable for broken $ISU(1|1)$ supersymmetry, as it was in [3]. Instead, it can be interpreted as a Nambu-Goldstone variable for the symmetry generated by $S$. As we now explain, this is the generator of a worldline supersymmetry, so the expansion of a wavefunction in $\xi$ is the ($ISU(1|1)$-invariant) expansion of a worldline superfield.

### 5.3 Worldline supersymmetry revisited

The anticommutation relation (5.45) implies that

$$\{S, S^\ddagger\} = 2\kappa H_{\rm susy}\,, \qquad H_{\rm susy} = H - 4\kappa M\,, \tag{5.50}$$

and one can similarly show that $\{S, S\} = 0 = \{S^\ddagger, S^\ddagger\}$. It is therefore natural to interpret $S$ as an $\mathcal{N}$=2 worldline supersymmetry charge, for Hamiltonian $H_{\rm susy}$, but the assumption that $M \le 0$ is crucial to this interpretation because $S^\ddagger$ is otherwise not the hermitian conjugate of $S$ with respect to a non-negative norm. Indeed, the anticommutator (5.45) would, if valid, imply that $N \ge 2M$, which would exclude states with $N<2M$ when $M>0$. As noted earlier, we must revert to the $G = 1$ norm when $N<2M$, in which case the anticommutator (5.45), and hence (5.50), is modified. One finds that

$$\{S, S^\ddagger\} = 2\kappa |H_{\rm susy}| \qquad (M>0)\,. \tag{5.51}$$

One could attempt to interpret this as a supersymmetry anticommutator with $|H_{\rm susy}|$ as a new Hamiltonian but this would be pointless as it does not imply worldline supersymmetry of the planar superflag model. For this reason, the planar superflag model has a hidden worldline supersymmetry only for $M \le 0$, so let us now assume that $M \le 0$.

A standard consequence of (5.50) is that $S$ and $S^\ddagger$ can only annihilate states that are annihilated by $H_{\rm susy}$, which are eigenstates of $H$ with energy $4\kappa M$. Given that $H_{an} = 2\kappa N_{an}$, for positive $\kappa$, and $M \le 0$, such states can exist only when $M$=0, in which case they are zero-energy states. In standard supersymmetric quantum mechanics, all zero-energy eigenstates must be annihilated by all supersymmetry charges. Our case is slightly different: a given zero-energy eigenstate need not be annihilated by $S$, or $S^\ddagger$, but if it is not then the resulting state has zero norm. This follows from the expressions (5.43) and (5.25), and the fact that $\Psi^{(0)}_{an} = \psi^{(0)}_{-}$ has zero norm at $M = 0$ (see (5.35)). Nevertheless, it is still true that all *physical* zero energy states are annihilated by both $S$ and $S^\ddagger$ because the physical subspace is spanned by equivalence classes of states modulo the addition of a zero-norm state. The number of these physical ground states is precisely half the total number of ground states, and hence non-zero, so the worldline supersymmetry is restored at $M$=0. This is expected from the equivalence with the superplane model, for which we know that the worldline supersymmetry is unbroken. In contrast, there are no supersymmetric ground states when $M<0$, so *worldline supersymmetry is spontaneously broken for* $M<0$.

These quantum results have classical analogs. To see this, we observe that the charges $S$ and $S^\ddagger$ generate transformations of the phase-space variables that leave invariant the phase-space form of the classical action, which is given in [3]. After solving the phase-space constraints and eliminating the momentum variables, one finds the infinitesimal transformation laws

$$\begin{aligned}
\delta z &= -\epsilon\xi\left(\dot z + \bar\xi\dot\zeta\right),\\
\delta\zeta &= = -\left[\left(1+\bar\xi\xi\right)\dot z + \bar\xi\dot\zeta\right]\bar\epsilon\,,\\
\delta\xi &= -2i\kappa\,\bar\epsilon\,,
\end{aligned} \tag{5.52}$$

where $\epsilon$ is the complex anticommuting parameter, with complex conjugate $\bar{\epsilon}$. The transformations of $(\bar{z}, \bar{\zeta}, \bar{\xi})$ are obtained by taking the complex conjugate. One may verify that these transformations leave invariant the classical Lagrangian (5.1), up to a total derivative, for any value of $M$, and have the same on-shell closure $\sim 2\kappa\partial_t$ for all relevant variables. Note that the transformations of $(z, \zeta)$ are on-shell equivalent to those of (4.34), and that they are compatible with the relation $\dot{\zeta} = -\xi\dot{z}$ since $\delta(-\dot{\zeta}/\dot{z}) = 2\kappa i\,\epsilon$, on shell.

Although the on-shell relation $\dot{\zeta} = -\xi\dot{z}$ suggests that $\xi$ is a 'composite' variable at $\dot{z} \neq 0$, it should now be recalled that it becomes an independent variable in a classical ground state corresponding to $\dot{z} = 0$, at least when $M < 0$. Then its inhomogeneous transformation implies that it is a Nambu-Goldstone variable for spontaneously broken worldline supersymmetry. Thus, classical supersymmetry is spontaneously broken when $M<0$. The $M=0$ case is different because, as mentioned earlier, $\xi$ can then be 'gauged away' in a classical ground state and the $\delta\xi$ transformation of (5.52) becomes just a particular case of the corresponding gauge transformation. So classical worldline supersymmetry is unbroken when $M=0$. This of course could be anticipated from the equivalence of the $M=0$ planar superflag model to the superplane model. The classical physics therefore parallels the quantum physics: supersymmetry is spontaneously broken for $M<0$ but restored at $M=0$.

## 6. Conclusions

Earlier studies of super-Landau models [1 − 3] concluded that these models have ghosts in all Landau levels but some number of the low-lying ones, as might be expected for a theory with 'higher-derivative' fermion kinetic terms, but this conclusion was grounded on a particular choice of (super)Hilbert space norm. Here we have investigated the possibility of other norms consistent with symmetries and hermiticity of the Hamiltonian. We have found that there is an alternative norm. This alternative norm is positive for the superplane Landau model, as is implicit in the previous work of Hasebe [8], and hence for the equivalent $M = 0$ planar superflag model. The alternative norm is also positive for the $M<0$ planar superflag Landau models, while for $M>0$ the positive norm is a 'dynamical' combination of both possibilities (in the sense that the choice depends on the level, and hence on the Hamiltonian). Thus, it is always possible to find a positive norm. However, this positive norm is not always positive-definite because there are zero-norm states when $2M$ is a non-negative integer.[13] This means that the positive norm cannot always be identified with the natural positive-definite norm in the basis for which the Hamiltonian is diagonal, so merely noting the existence of the latter is not sufficient, in general, to 'exorcize' the super-Landau ghosts; the detailed analysis performed here was necessary.

The possibility of modifying the Hilbert space norm in order to convert an apparently unphysical quantum theory into a physical one is the underlying theme of '$\mathcal{PT}$-symmetric' quantum theory, and a number of methods for investigating these possibilities have been developed in this context. Here we have taken over these methods, extending them to

[13] This phenomenon was shown in [3] to be associated with a fermionic gauge-invariance.

models with anticommuting variables. Apart from the basic point that one may be able to adjust the (super)Hilbert space norm, via a 'metric operator' so as to achieve a positive inner product, the main consequence of a redefined norm is a redefined notion of hermitian conjugation. Specifically, operators that do not commute with the metric operator have hermitian conjugates that do not coincide with the naive conjugate. For the planar super-Landau models investigated here, we found that the supersymmetry charges have hermitian conjugates that are shifted, relative to their naive hermitian conjugates, by conserved 'shift' operators.

Remarkably, these 'shift' operators are worldline supersymmetry charges, analogous to those noted by Hasebe in his version of the superplane Landau model [8]. Classically, these charges generate transformations of the variables that leave the Lagrangian invariant, up to a total derivative, and the classical ground state solution is supersymmetric; this feature is maintained in the quantum theory, since the quantum ground state is annihilated by the quantum supersymmetry charges. Although the worldline supersymmetry algebra is the standard one of supersymmetric quantum mechanics, the form of the supersymmetry transformations is non-standard. It would be interesting to see whether there is a superspace version of the model that makes manifest the invariance of the classical action.

The 'hidden' worldline supersymmetry of planar super-Landau models emerges most naturally in the more general planar superflag models when $M<0$ because the additional anticommuting parameter of these models then has a natural interpretation as the Nambu-Goldstone variable for a broken worldline supersymmetry. Quantum mechanically, the worldline supersymmetry is spontaneously broken because the supersymmetry charges fail to annihilate the ground state. One might be tempted to conclude that the Witten index is therefore zero and that, as a consequence, the worldline supersymmetry will remain spontaneously broken for any (non-positive) value of the parameter $M$ (since quantum corrections to models of supersymmetric quantum mechanics generally raise the energy of any state that would otherwise 'accidentally' have zero energy). However, this conclusion would not be correct; not because quantum corrections fail to raise the energy of an otherwise zero-energy state but because the spectrum changes discontinuously at $M=0$ due to the vanishing of the norm of half the lowest Landau level states. Thus, worldline supersymmetry is restored at $M=0$ by a novel mechanism.

The discontinuity in the spectrum at $M=0$ suggests that the Witten index is discontinuous too, but the infinite degeneracy of the lowest Landau level in planar Landau models may instead mean that the index is ill-defined. For this reason, among others, it would be interesting to know what happens for the spherical super-Landau models, for which the degeneracies at each level are finite. Of course, the issue arises only if the spherical super-Landau models also exhibit a 'hidden' worldline supersymmetry, and it remains to be seen whether this is the case.

## Acknowledgments

T.C. and L.M. thank the Institute for Advanced Study for its hospitality and support, and for providing a stimulating environment in which part of this work was completed. We thank Carl Bender, Shanta de Alwis, Jaume Gomis, Jurek Lukierski, Robert Myers, Andrei Smilga, Juan Maldacena and Dima Sorokin for useful discussions. This material is based upon work supported by the National Science Foundation under Grants No's. 0303550 and 0555603. E.I. acknowledges a partial support from the Grant RFBR 06-02-16684, RFBR-DFG Grant 06-02-04012 and the Grant INTAS-05-7928. P.K.T. thanks the EPSRC for financial support.

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